# Calculus of Variations

The biggest step from derivatives with one variable to derivatives with many variables is from one to two. After that, going from two to three was just more algebra and more complicated pictures. Now the step will be from a finite number of variables to an infinite number. That will require a new set of tools, yet in many ways the techniques are not very different from those you know.

If you've never read chapter 19 of volume II of the Feynman Lectures in Physics, now would be a good time. It's a classic introduction to the area. For a deeper look at the subject, pick up MacCluer's book referred to in the Bibliography at the beginning of this book.

## 16.1 Examples

What line provides the shortest distance between two points? A straight line of course, no surprise there. But not so fast, with a few twists on the question the result won't be nearly as obvious. How do I measure the length of a curved (or even straight) line? Typically with a ruler. For the curved line I have to do successive approximations, breaking the curve into small pieces and adding the finite number of lengths, eventually taking a limit to express the answer as an integral. Even with a straight line I will do the same thing if my ruler isn't long enough.

Put this in terms of how you do the measurement: Go to a local store and purchase a ruler. It's made out of some real material, say brass. The curve you're measuring has been laid out on the ground, and you move along it, counting the number of times that you use the ruler to go from one point on the curve to another. If the ruler measures in decimeters and you lay it down 100 times along the curve, you have your first estimate for the length, 10.0 meters. Do it again, but use a centimeter length and you need 1008 such lengths: 10.08 meters.

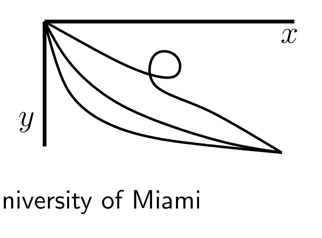
That's tedious, but simple. Now do it again for another curve and compare their lengths. Here comes the twist: The ground is not at a uniform temperature. Perhaps you're making these measurements over a not-fully-cooled lava flow in Hawaii. Brass will expand when you heat it, so if the curve whose length you're measuring passes over a hot spot, then the ruler will expand when you place it down, and you will need to place it down fewer times to get to the end of the curve. You will measure the curve as shorter. Now it is not so clear which curve will have the shortest (measured) length. If you take the straight line and push it over so that it passes through a hotter region, then you may get a smaller result.

Let the coefficient of expansion of the ruler be $\alpha$, assumed constant. For modest temperature changes, the length of the ruler is $\ell' = (1+\alpha\Delta T)\ell$. The length of a curve as measured with this ruler is

$$\int \frac{d\ell}{1+\alpha T} \tag{16.1}$$

Here I'm taking $T = 0$ as the base temperature for the ruler and $d\ell$ is the length you would use if everything stayed at this temperature. With this measure for length, it becomes an interesting problem to discover which path has the shortest "length." The formal term for the path of shortest length is geodesic.

In section 13.1 you saw integrals that looked very much like this, though applied to a different problem. There I looked at the time it takes a particle to slide down a curve under gravity. That time is the integral of $dt = d\ell/v$, where $v$ is the particle's speed, a function of position along the path. Using conservation of energy, the expression for the time to slide down a curve was Eq. (13.6).




$$\int dt = \int \frac{d\ell}{\sqrt{(2E/m)+2gy}} \tag{16.2}$$

In that chapter I didn't attempt to answer the question about which curve provides the quickest route to the end, but in this chapter I will. Even qualitatively you can see a parallel between these two problems. You get a shorter length by pushing the curve into a region of higher temperature. You get a shorter time by pushing the curve lower, (larger $y$). In the latter case, this means that you drop fast to pick up speed quickly. In both cases the denominator in the integral is larger. You can overdo it of course. Push the curve too far and the value of $\int d\ell$ itself can become too big. It's a balance.

In problems 2.35 and 2.39 you looked at the amount of time it takes light to travel from one point to another along various paths. Is the time a minimum, a maximum, or neither? In these special cases, you saw that this is related to the focus of the lens or of the mirror. This is a very general property of optical systems, and is an extension of some of the ideas in the first two examples above.

These questions are sometimes pretty and elegant, but are they related to anything else? Yes. Newton's classical mechanics can be reformulated in this language and it leads to powerful methods to set up the equations of motion in complicated problems. The same ideas lead to useful approximation techniques in electromagnetism, allowing you to obtain high-accuracy solutions to problems for which there is no solution by other means.

### 16.2 Functional Derivatives

It is time to get specific and to implement* these concepts. All the preceding examples can be expressed in the same general form. In a standard $x$-$y$ rectangular coordinate system,

$$d\ell = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\,\sqrt{1 + y'^2}$$

Then Eq. (16.1) is

$$\int_a^b dx\,\frac{\sqrt{1 + y'^2}}{1 + \alpha T(x, y)} \tag{16.3}$$

This measured length depends on the path, and I've assumed that I can express the path with $y$ as a function of $x$. No loops. You can allow more general paths by using another parametrization: $x(t)$ and $y(t)$. Then the same integral becomes

$$\int_{t_1}^{t_2} dt\,\frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{1 + \alpha T\big(x(t), y(t)\big)} \tag{16.4}$$

The equation (16.2) has the same form

$$\int_a^b dx\,\frac{\sqrt{1 + y'^2}}{\sqrt{(2E/m) + 2gy}}$$

And the travel time for light through an optical system is

$$\int dt = \int \frac{d\ell}{v} = \int_a^b dx\,\frac{\sqrt{1 + y'^2}}{v(x, y)}$$

where the speed of light is some known function of the position.

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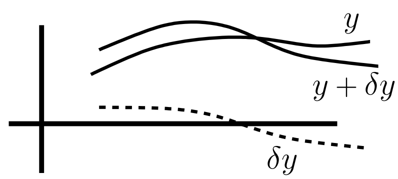
* If you find the methods used in this section confusing, you may prefer to look at an alternate approach to the subject as described in section 16.6. Then return here.

In all of these cases the output of the integral depends on the path taken. It is a *functional* of the path, a scalar-valued function of a function variable. Denote the argument by square brackets.

$$I[y] = \int_a^b dx\, F\big(x, y(x), y'(x)\big) \tag{16.5}$$

The specific $F$ varies from problem to problem, but the preceding examples all have this general form, even when expressed in the parametrized variables of Eq. (16.4).

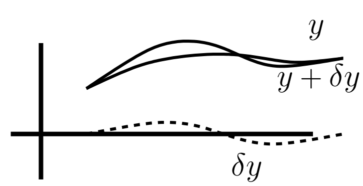
The idea of differential calculus is that you can get information about a function if you try changing the independent variable by a small amount. Do the same thing here. Now however the independent variable is the whole path, so I'll change that path by some small amount and see what happens to the value of the integral $I$. This approach to the subject is due to Lagrange. The development in section 16.6 comes from Euler.




$$\begin{aligned} \Delta I &= I[y+\delta y] - I[y] \\ &= \int_{a+\Delta a}^{b+\Delta b} dx\, F\big(x, y(x)+\delta y(x), y'(x)+\delta y'(x)\big) - \int_a^b dx\, F\big(x, y(x), y'(x)\big) \end{aligned} \tag{16.6}$$

The (small) function $\delta y(x)$ is the vertical displacement of the path in this coordinate system. To keep life simple for the first attack on this problem, I'll take the special case for which the endpoints of the path are fixed. That is,

$$\Delta a = 0, \qquad \Delta b = 0, \qquad \delta y(a) = 0, \qquad \delta y(b) = 0$$




To compute the value of Eq. (16.6) use the power series expansion of $F$, as in section 2.5.

$$\begin{aligned} F(x+\Delta x, y+\Delta y, z+\Delta z) = F(x,y,z) &+ \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y + \frac{\partial F}{\partial z}\Delta z \\ &+ \frac{\partial^2 F}{\partial x^2}\frac{(\Delta x)^2}{2} + \frac{\partial^2 F}{\partial x \partial y}\Delta x\, \Delta y + \cdots \end{aligned}$$

For now look at just the lowest order terms, linear in the changes, so ignore the second order terms. In this application, there is no $\Delta x$.

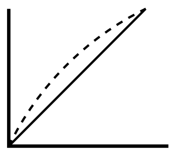
$$F(x, y+\delta y, y'+\delta y') = F(x,y,y') + \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'$$

plus terms of higher order in $\delta y$ and $\delta y'$.

Put this into Eq. (16.6), and

$$\delta I = \int_a^b dx\, \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right] \tag{16.7}$$

For example, Let $F = x^2 + y^2 + y'^{\,2}$ on the interval $0 \le x \le 1$. Take a base path to be a straight line from $(0,0)$ to $(1,1)$. Choose for the change in the path $\delta y(x) = \epsilon x(1-x)$. This is simple and it satisfies the boundary conditions.



$$\begin{aligned} I[y] &= \int_0^1 dx\,\left[x^2 + y^2 + y'^2\right] = \int_0^1 dx\,\left[x^2 + x^2 + 1^2\right] = \frac{5}{3} \\ I[y+\delta y] &= \int_0^1 \left[x^2 + \big(x + \epsilon x(1-x)\big)^2 + \big(1 + \epsilon(1-2x)\big)^2\right] \\ &= \frac{5}{3} + \frac{1}{6}\epsilon + \frac{11}{30}\epsilon^2 \end{aligned} \tag{16.8}$$

The value of Eq. (16.7) is

$$\delta I = \int_0^1 dx\,\left[2y\delta y + 2y'\delta y'\right] = \int_0^1 dx\,\left[2x\epsilon x(1-x) + 2\epsilon(1-2x)\right] = \frac{1}{6}\epsilon$$

Return to the general case of Eq. (16.7) and you will see that I've explicitly used only *part* of the assumption that the endpoint of the path hasn't moved, $\Delta a = \Delta b = 0$. There's nothing in the body of the integral itself that constrains the change in the $y$-direction, and I had to choose the function $\delta y$ by hand so that this constraint held. In order to use the equations $\delta y(a) = \delta y(b) = 0$ more generally, there is a standard trick: integrate by parts. You'll *always* integrate by parts in these calculations.

$$\int_a^b dx\,\frac{\partial F}{\partial y'}\delta y' = \int_a^b dx\,\frac{\partial F}{\partial y'}\frac{d\,\delta y}{dx} = \frac{\partial F}{\partial y'}\delta y\bigg|_a^b - \int_a^b dx\,\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta y(x)$$

This expression allows you to use the information that the path hasn't moved at its endpoints in the $y$ direction either. The boundary term from this partial integration is

$$\frac{\partial F}{\partial y'}\delta y\bigg|_a^b = \frac{\partial F}{\partial y'}\big(b, y(b)\big)\delta y(b) - \frac{\partial F}{\partial y'}\big(a, y(a)\big)\delta y(a) = 0$$

Put the last two equations back into the expression for $\delta I$, Eq. (16.7) and the result is

$$\delta I = \int_a^b dx\,\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\delta y \tag{16.9}$$

Use this expression for the same example $F = x^2 + y^2 + y'^{\,2}$ with $y(x) = x$ and you have

$$\delta I = \int_0^1 dx\,\left[2y - \frac{d}{dx}2y'\right]\delta y = \int_0^1 dx\,\left[2x - 0\right]\epsilon x(1-x) = \frac{1}{6}\epsilon$$

This is sort of like Eq. (8.16),

$$df = \vec G \cdot d\vec r = \operatorname{grad} f \cdot d\vec r = \nabla f \cdot d\vec r = \frac{\partial f}{\partial x_k}dx_k = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots$$

The differential change in the function depends linearly on the change $d\vec r$ in the coordinates. It is a sum over the terms with $dx_1$, $dx_2$, .... This is a precise parallel to Eq. (16.9), except that the sum over discrete index $k$ is now an integral over the continuous index $x$. The change in $I$ is a linear functional of

the change $\delta y$ in the independent variable $y$; this $\delta y$ corresponds to the change $d\vec r$ in the independent variable $\vec r$ in the other case. The coefficient of the change, instead of being called the gradient, is called the "functional derivative" though it's essentially the same thing.

$$\frac{\delta I}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right), \qquad \delta I[y,\delta y] = \int dx\, \frac{\delta I}{\delta y}\big(x, y(x), y'(x)\big)\,\delta y(x) \tag{16.10}$$

and for a change, I've indicated explicitly the dependence of $\delta I$ on the two functions $y$ and $\delta y$. This parallels the equation (8.13). The statement that this functional derivative vanishes is called the Euler-Lagrange equation.

Return to the example $F = x^2 + y^2 + y'^2$, then

$$\frac{\delta I}{\delta y} = \frac{\delta}{\delta y}\int_0^1 dx\; \left[x^2 + y^2 + y'^2\right] = 2y - \frac{d}{dx}2y' = 2y - 2y''$$

What is the minimum value of $I$? Set this derivative to zero.

$$y'' - y = 0 \implies y(x) = A\cosh x + B\sinh x$$

The boundary conditions $y(0) = 0$ and $y(1) = 1$ imply $y = B\sinh x$ where $B = 1/\sinh 1$. The value of $I$ at this point is

$$I[B\sinh x] = \int_0^1 dx\; \left[x^2 + B^2\sinh^2 x + B^2\cosh^2 x\right] = \frac{1}{3} + \coth 1 \tag{16.11}$$

Is it a minimum? Yes, but just as with the ordinary derivative, you have to look at the next order terms to determine that. Compare this value of $I[y] = 1.64637$ to the value $5/3$ found for the nearby function $y(x) = x$, evaluated in Eq. (16.8).

Return to one of the examples in the introduction. What is the shortest distance between two points, but for now assume that there's no temperature variation. Write the length of a path for a function $y$ between fixed endpoints, take the derivative, and set that equal to zero.

$$L[y] = \int_a^b dx\,\sqrt{1+y'^2}, \qquad \text{so}$$

$$\frac{\delta L}{\delta y} = -\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = -\frac{y''}{\sqrt{1+y'^2}} + \frac{y'^2\, y''}{\left(1+y'^2\right)^{3/2}} = \frac{-y''}{\left(1+y'^2\right)^{3/2}} = 0$$

For a minimum length then, $y'' = 0$, and that's a straight line. Surprise!

Do you really have to work through this mildly messy manipulation? Sometimes, but not here. Just notice that the derivative is in the form

$$\frac{\delta L}{\delta y} = -\frac{d}{dx}f(y') = 0 \tag{16.12}$$

so it doesn't matter what the particular $f$ is and you get a straight line. $f(y')$ is a constant so $y'$ must be constant too. Not so fast! See section 16.9 for another opinion.

**16.3 Brachistochrone**

Now for a tougher example, again from the introduction. In Eq. (16.2), which of all the paths between fixed initial and final points provides the path of least time for a particle sliding along it under gravity. Such a path is called a brachistochrone. This problem was first proposed by Bernoulli (one of them), and was solved by several people including Newton, though it's unlikely that he used the methods developed here, as the general structure we now call the calculus of variations was decades in the future.

Assume that the particle starts from rest so that $E = 0$, then

$$T[y] = \int_0^{x_0} dx\, \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} \tag{16.13}$$

For the minimum time, compute the derivative and set it to zero.

$$\sqrt{2g}\,\frac{\delta T}{\delta y} = -\frac{\sqrt{1 + y'^2}}{2y^{3/2}} - \frac{d}{dx}\frac{y'}{2\sqrt{y}\sqrt{1 + y'^2}} = 0$$

This is starting to look intimidating, leading to an impossibly* messy differential equation. Is there another way? Yes. Why must $x$ be the independent variable? What about using $y$? In the general setup leading to Eq. (16.10) nothing changes except the symbols, and you have

$$I[x] = \int dy\, F(y, x, x') \;\longrightarrow\; \frac{\delta I}{\delta x} = \frac{\partial F}{\partial x} - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) \tag{16.14}$$

Equation (16.13) becomes

$$T[x] = \int_0^{y_0} dy\, \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} \tag{16.15}$$

The function $x$ does not appear explicitly in this integral, just its derivative $x' = dx/dy$. This simplifies the functional derivative, and the minimum time now comes from the equation

$$\frac{\delta I}{\delta x} = 0 - \frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) = 0 \tag{16.16}$$

This is much easier. $d()/dy = 0$ means that the object in parentheses is a constant.

$$\frac{\partial F}{\partial x'} = C = \frac{1}{\sqrt{2gy}}\frac{x'}{\sqrt{1 + x'^2}}$$

Solve this for $x'$ and you get (let $K = C\sqrt{2g}$)

$$x' = \frac{dx}{dy} = \sqrt{\frac{K^2 y}{1 - K^2 y}}, \qquad \text{so} \qquad x(y) = \int dy\, \sqrt{\frac{K^2 y}{1 - K^2 y}}$$

This is an elementary integral. Let $2a = 1/K^2$

$$x(y) = \int dy\, \frac{y}{\sqrt{2ay - y^2}} = \int dy\, \frac{y}{\sqrt{a^2 - a^2 + 2ay - y^2}} = \int dy\, \frac{(y - a) + a}{\sqrt{a^2 - (y - a)^2}}$$

* Only improbably. See problem 16.12.

Make the substitution $(y-a)^2 = z$ in the first half of the integral and $(y-a) = a\sin\theta$ in the second half.

$$x(y) = \frac{1}{2}\int dz\,\frac{1}{\sqrt{a^2-z}} + \int \frac{a^2\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$
$$= -\sqrt{a^2-z} + a\theta = -\sqrt{a^2-(y-a)^2} + a\sin^{-1}\left(\frac{y-a}{a}\right) + C'$$

The boundary condition that $x(0)=0$ determines $C' = a\pi/2$, and the other end of the curve determines $a$: $x(y_0) = x_0$. You can rewrite this as

$$x(y) = -\sqrt{2ay-y^2} + a\cos^{-1}\left(\frac{a-y}{a}\right) \tag{16.17}$$

This is a cycloid. What's a cycloid and why does this equation describe one? See problem 16.2.

**x-independent**

In Eqs. (16.15) and (16.16) there was a special case for which the dependent variable was missing from $F$. That made the equation much simpler. What if the independent variable is missing? Does that provide a comparable simplification? Yes, but it's trickier to find.

$$I[y] = \int dx\,F(x,y,y') \longrightarrow \frac{\delta I}{\delta y} = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{16.18}$$

Use the chain rule to differentiate $F$ with respect to $x$.

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{dy}{dx}\frac{\partial F}{\partial y} + \frac{dy'}{dx}\frac{\partial F}{\partial y'} \tag{16.19}$$

Multiply the Lagrange equation (16.18) by $y'$ to get

$$y'\frac{\partial F}{\partial y} - y'\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Now substitute the term $y'(\partial F/\partial y)$ from the preceding equation (16.19).

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{dy'}{dx}\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

The last two terms are the derivative of a product.

$$\frac{dF}{dx} - \frac{\partial F}{\partial x} - \frac{d}{dx}\left[y'\frac{\partial F}{\partial y'}\right] = 0 \tag{16.20}$$

If the function $F$ has no explicit $x$ in it, the second term is zero, and the equation is now a derivative

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0 \qquad \text{and} \qquad y'\frac{\partial F}{\partial y'} - F = C \tag{16.21}$$

This is already down to a first order differential equation. The combination $y'F_{y'} - F$ that appears on the left of the second equation is important. It's called the Hamiltonian.

## 16.4 Fermat's Principle

Fermat's principle of least time provides a formulation of geometrical optics. When you don't know about the wave nature of light, or if you ignore that aspect of the subject, it seems that light travels in straight lines — at least until it hits something. Of course this isn't fully correct, because when light hits a piece of glass it refracts and its path bends and follows Snell's equation. All of these properties of light can be described by Fermat's statement that the path light follows will be that path that takes the least* time.

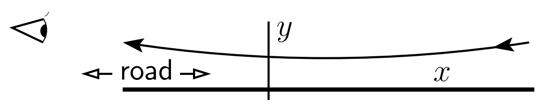
$$T = \int dt = \int \frac{d\ell}{v} = \frac{1}{c}\int n\,d\ell \tag{16.22}$$

The total travel time is the integral of the distance $d\ell$ over the speed (itself a function of position). The index of refraction is $n = c/v$, where $c$ is the speed of light in vacuum, so I can rewrite the travel time in the above form using $n$. The integral $\int n\,d\ell$ is called the optical path.

From this idea it is very easy to derive the rule for reflection at a surface: angle of incidence equals angle of reflection. It is equally easy to derive Snell's law. (See problem 16.5.) I'll look at an example that would be difficult to do by any means *other* than Fermat's principle: Do you remember what an asphalt road looks like on a very hot day? If you are driving a car on a black colored road you may see the road in the distance appear to be covered by what looks like water. It has a sort of mirror-like sheen that is always receding from you — the "hot road mirage". You can never catch up to it. This happens because the road is very hot and it heats the air next to it, causing a strong temperature gradient near the surface. The density of the air decreases with rising temperature because the pressure is constant. That in turn means that the index of refraction will decrease with the rising temperature near the surface. The index will then be an increasing function of the distance above ground level, $n = f(y)$, and the travel time of light will depend on the path taken.

$$\int n\,d\ell = \int f(y)\,d\ell = \int f(y)\sqrt{1+x'^2}\,dy = \int f(y)\sqrt{1+y'^2}\,dx \tag{16.23}$$

What is $f(y)$? I'll leave that for a moment and then after carrying the calculation through for a while I can pick an $f$ that is both plausible and easy to manipulate.




Should $x$ be the independent variable, or $y$? Either should work, and I chose $y$ because it seemed likely to be easier. (See problem 16.6 however.) The integrand then does not contain the dependent variable $x$.

$$\text{minimize} \quad \int n\,d\ell = \int f(y)\sqrt{1+x'^2}\,dy \implies \frac{d}{dy}\frac{\partial}{\partial x'}\left[f(y)\sqrt{1+x'^2}\right] = 0$$

$$f(y)\frac{x'}{\sqrt{1+x'^2}} = C$$

Solve for $x'$ to get

$$f(y)^2 x'^2 = C^2\left(1+x'^2\right) \qquad \text{so} \qquad x' = \frac{dx}{dy} = \frac{C}{\sqrt{f(y)^2 - C^2}} \tag{16.24}$$

* Not always least. This just requires the first derivative to be zero; the second derivative is addressed in section 16.10. "Fermat's principle of stationary time" may be more accurate, but "Fermat's principle of least time" is entrenched in the literature.

At this point pick a form for the index of refraction that will make the integral easy and will still plausibly represent reality. The index increases gradually above the road surface, and the simplest function works: $f(y) = n_0(1 + \alpha y)$. The index increases linearly above the surface.

$$x(y) = \int \frac{C}{\sqrt{n_0^2(1+\alpha y)^2 - C^2}}\,dy = \frac{C}{\alpha n_0}\int dy \frac{1}{\sqrt{(y+1/\alpha)^2 - C^2/\alpha^2 n_0^2}}$$

This is an elementary integral. Let $u = y + 1/\alpha$, then $u = (C/\alpha n_0)\cosh\theta$.

$$x = \frac{C}{\alpha n_0}\int d\theta \;\Rightarrow\; \theta = \frac{\alpha n_0}{C}(x - x_0) \;\Rightarrow\; y = -\frac{1}{\alpha} + \frac{C}{\alpha n_0}\cosh\big((\alpha n_0/C)(x - x_0)\big)$$

$C$ and $x_0$ are arbitrary constants, and $x_0$ is obviously the center of symmetry of the path. You can relate the other constant to the $y$-coordinate at that same point: $C = n_0(\alpha y_0 + 1)$.

Because the value of $\alpha$ is small for any real roadway, look at the series expansion of this hyperbolic function to the lowest order in $\alpha$.

$$y \approx y_0 + \alpha(x - x_0)^2/2 \tag{16.25}$$

When you look down at the road you can be looking at an image of the sky. The light comes from the sky near the horizon down toward the road at an angle of only a degree or two. It then curves up so that it can enter your eye as you look along the road. The shimmering surface is a reflection of the distant sky or in this case an automobile — a mirage.

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## 16.5 Electric Fields

The energy density in an electric field is $\epsilon_0 E^2/2$. For the static case, this electric field is the gradient of a potential, $\vec{E} = -\nabla\phi$. Its total energy in a volume is then

$$W = \frac{\epsilon_0}{2}\int dV\,(\nabla\phi)^2 \tag{16.26}$$

What is the minimum value of this energy? Zero of course, if $\phi$ is a constant. That question is too loosely stated to be much use, but keep with it for a while and it will be possible to turn it into something more precise and more useful. As with any other derivative taken to find a minimum, change the independent variable by a small amount. This time the variable is the function $\phi$, so really this quantity $W$ can more fully be written as a functional $W[\phi]$ to indicate its dependence on the potential function.

$$\begin{aligned} W[\phi+\delta\phi] - W[\phi] &= \frac{\epsilon_0}{2}\int dV\,\big(\nabla(\phi+\delta\phi)\big)^2 - \frac{\epsilon_0}{2}\int dV\,(\nabla\phi)^2 \\ &= \frac{\epsilon_0}{2}\int dV\,\big(2\nabla\phi\cdot\nabla\delta\phi + (\nabla\delta\phi)^2\big) \end{aligned}$$

* Donald Collins, Warren Wilson College

Now pull out a vector identity from problem 9.36,

$$\nabla\cdot(f\vec g)=\nabla f\cdot\vec g+f\nabla\cdot\vec g$$

and apply it to the previous line with $f=\delta\phi$ and $\vec g=\nabla\phi$.

$$W[\phi+\delta\phi]-W[\phi]=\epsilon_0\int dV\ \left[\nabla\cdot(\delta\phi\nabla\phi)-\delta\phi\nabla^2\phi\right]+\frac{\epsilon_0}{2}\int dV\,(\nabla\delta\phi)^2$$

The divergence term is set up to use Gauss's theorem; this is the vector version of integration by parts.

$$W[\phi+\delta\phi]-W[\phi]=\epsilon_0\oint d\vec A\cdot(\nabla\phi)\delta\phi-\epsilon_0\int dV\,\delta\phi\nabla^2\phi+\frac{\epsilon_0}{2}\int dV\,(\nabla\delta\phi)^2 \qquad (16.27)$$

If the value of the potential $\phi$ is specified everywhere on the boundary, then I'm not allowed to change it in the process of finding the change in $W$. That means that $\delta\phi$ vanishes on the boundary. That makes the boundary term, the $d\vec A$ integral, vanish. Its integrand is zero everywhere on the surface of integration.

In looking for a minimum energy I want to set the first derivative to zero, and that's the coefficient of the term linear in $\delta\phi$.

$$\frac{\delta W}{\delta\phi}=-\epsilon_0\nabla^2\phi=0$$

The function that produces the minimum value of the total energy (with these fixed boundary conditions) is the one that satisfies Laplace's equation. Is it really a minimum? Yes. In this instance it's very easy to see that. The extra term in the change of $W$ is $\int dV\,(\nabla\delta\phi)^2$. That is positive no matter what $\delta\phi$ is.

That the correct potential function is the one having the minimum energy allows for an efficient approximate method to solve electrostatic problems. I'll illustrate this using an example borrowed from the Feynman Lectures in Physics and that you can also solve exactly: What is the capacitance of a length of coaxial cable? (Neglect the edge effects at the ends of the cable of course.) Let the inner and outer radii be $a$ and $b$, and the length $L$. A charge density $\lambda$ is on the inner conductor (and therefore $-\lambda$ on the inside of the outer conductor). It creates a radial electric field of size $\lambda/2\pi\epsilon_0 r$. The potential difference between the conductors is

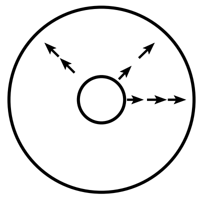
$$\Delta V=\int_a^b dr\,\frac{\lambda}{2\pi\epsilon_0 r}=\frac{\lambda\ln(b/a)}{2\pi\epsilon_0} \qquad (16.28)$$

The charge on the inner conductor is $\lambda L$, so $C=Q/\Delta V=2\pi\epsilon_0 L/\ln(b/a)$, where $\Delta V=V_b-V_a$.

The total energy satisfies $W=C\Delta V^2/2$, so for the given potential difference, knowing the energy is the same as knowing the capacitance.

This exact solution provides a laboratory to test the efficacy of a variational approximation for the same problem. The idea behind this method is that you assume a form for the solution $\phi(r)$. This assumed form must satisfy the boundary conditions, but it need not satisfy Laplace's equation. It should also have one or more free parameters, the more the better up to the limits of your patience. Now compute the total energy that this trial function implies and then use the free parameters to minimize it. This function with minimum energy is the best approximation to the correct potential among all those with your assumed trial function.

Let the potential at $r = a$ be $V_a$ and at $r = b$ it is $V_b$. An example function that satisfies these conditions is



$$\phi(r) = V_a + (V_b - V_a)\frac{r-a}{b-a} \tag{16.29}$$

The electric field implied by this is $\vec{E} = -\nabla\phi = \hat{r}\,(V_a - V_b)/(b-a)$, a constant radial component. From (16.26), the energy is

$$\frac{\epsilon_0}{2}\int_a^b L\,2\pi r\,dr\left(\frac{d\phi}{dr}\right)^2 = \frac{\epsilon_0}{2}\int_a^b L\,2\pi r\,dr\left(\frac{V_b - V_a}{b-a}\right)^2 = \frac{1}{2}\pi L\epsilon_0\frac{b+a}{b-a}\Delta V^2$$

Set this to $C\Delta V^2/2$ to get $C$ and you have

$$C_{\text{approx}} = \pi L\epsilon_0\frac{b+a}{b-a}$$

How does this compare to the exact answer, $2\pi\epsilon_0 L/\ln(b/a)$? Let $x = b/a$.

$$\frac{C_{\text{approx}}}{C} = \frac{1}{2}\frac{b+a}{b-a}\ln(b/a) = \frac{1}{2}\frac{x+1}{x-1}\ln x$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$: | 1.1 | 1.2 | 1.5 | 2.0 | 3.0 | 10.0 |
| ratio: | 1.0007 | 1.003 | 1.014 | 1.04 | 1.10 | 1.41 |

Assuming a constant magnitude electric field in the region between the two cylinders is clearly not correct, but this estimate of the capacitance gives a remarkable good result even when the ratio of the radii is two or three. This is true even though I didn't even put in a parameter with which to minimize the energy. How much better will the result be if I do?

Instead of a linear approximation for the potential, use a quadratic.

$$\phi(r) = V_a + \alpha(r-a) + \beta(r-a)^2, \qquad \text{with} \qquad \phi(b) = V_b$$

Solve for $\alpha$ in terms of $\beta$ and you have, after a little manipulation,

$$\phi(r) = V_a + \Delta V\frac{r-a}{b-a} + \beta(r-a)(r-b) \tag{16.30}$$

Compute the energy from this.

$$W = \frac{\epsilon_0}{2}\int_a^b L\,2\pi r\,dr\left[\frac{\Delta V}{b-a} + \beta(2r-a-b)\right]^2$$

Rearrange this for easier manipulation by defining $2\beta = \gamma\Delta V/(b-a)$ and $c = (a+b)/2$ then

$$\begin{aligned} W &= \frac{\epsilon_0}{2}2L\pi\left(\frac{\Delta V}{b-a}\right)^2\int\big((r-c)+c\big)dr\big[1+\gamma(r-c)\big]^2 \\ &= \frac{\epsilon_0}{2}2L\pi\left(\frac{\Delta V}{b-a}\right)^2\big[c(b-a) + \gamma(b-a)^3/6 + c\gamma^2(b-a)^3/12\big] \end{aligned}$$

$\gamma$ is a free parameter in this calculation, replacing the original $\beta$. To minimize this energy, set the derivative $dW/d\gamma = 0$, resulting in the value $\gamma = -1/c$. At this value of $\gamma$ the energy is

$$W = \frac{\epsilon_0}{2} 2L\pi \left(\frac{\Delta V}{b-a}\right)^2 \left[\frac{1}{2}(b^2 - a^2) - \frac{(b-a)^3}{6(b+a)}\right] \tag{16.31}$$

Except for the factor of $\Delta V^2/2$ this is the new estimate of the capacitance, and to see how good it is, again take the ratio of this estimate to the exact value and let $x = b/a$.

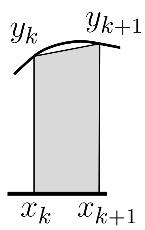
$$\frac{C'_{\text{approx}}}{C} = \ln x \,\frac{1}{2}\frac{x+1}{x-1}\left[1 - \frac{(x-1)^2}{3(x+1)^2}\right] \tag{16.32}$$

| $x$: | 1.1 | 1.2 | 1.5 | 2.0 | 3.0 | 10.0 |
|---|---|---|---|---|---|---|
| ratio: | 1.00000046 | 1.000006 | 1.00015 | 1.0012 | 1.0071 | 1.093 |

For only a one parameter adjustment, this provides very high accuracy. This sort of technique is the basis for many similar procedures in this and other contexts, especially in quantum mechanics.

## 16.6 Discrete Version

There is another way to find the functional derivative, one that more closely parallels the ordinary partial derivative. It is due to Euler, and he found it first, before Lagrange's discovery of the treatment that I've spent all this time on. Euler's method is perhaps more intuitive than Lagrange's, but it is not as easy to extend it to more than one dimension and it doesn't easily lend itself to the powerful manipulative tools that Lagrange's method does. This alternate method starts by noting that an integral is the limit of a sum. Go back to the sum and perform the derivative on *it*, finally taking a limit to get an integral. This turns the problem into a finite-dimensional one with ordinary partial derivatives. You have to decide which form of numerical integration to use, and I'll pick the trapezoidal rule, Eq. (11.15), with a constant interval. Other choices work too, but I think this entails less fuss. You don't see this approach as often as Lagrange's because it is harder to manipulate the equations with Euler's method, and the notation can become quite cumbersome. The trapezoidal rule for an integral is just the following picture, and all that you have to handle with any care are the endpoints of the integral.




$$x_k = a + k\Delta, \quad 0 \le k \le N \qquad \text{where} \qquad \frac{b-a}{N} = \Delta$$

$$\int_a^b dx\, f(x) = \lim \left[\frac{1}{2}f(a) + \sum_1^{N-1} f(x_k) + \frac{1}{2}f(b)\right]\Delta$$

The integral Eq. (16.5) involves $y'$, so in the same spirit, approximate this by the centered difference.

$$y'_k = y'(x_k) \approx \big(y(x_{k+1}) - y(x_{k-1})\big)/2\Delta$$

This evaluates the derivative *at* each of the coordinates $\{x_k\}$ instead of between them.

The discrete form of (16.5) is now

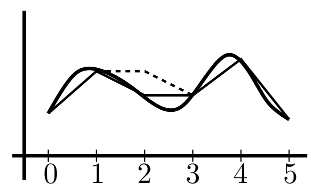
$$\begin{aligned} I_{\text{discrete}} &= \frac{\Delta}{2} F\big(a, y(a), y'(a)\big) \\ &+ \sum_1^{N-1} F\big(x_k, y(x_k), (y(x_{k+1}) - y(x_{k-1}))/2\Delta\big)\Delta + \frac{\Delta}{2}F\big(b, y(b), y'(b)\big) \end{aligned}$$

Not quite yet. What about $y'(a)$ and $y'(b)$? The endpoints $y(a)$ and $y(b)$ aren't changing, but that doesn't mean that the slope there is fixed. At these two points, I can't use the centered difference scheme for the derivative, I'll have to use an asymmetric form to give

$$I_{\text{discrete}} = \frac{\Delta}{2}F\big(a, y(a), (y(x_1) - y(x_0))/\Delta\big) + \frac{\Delta}{2}F\big(b, y(b), (y(x_N) - y(x_{N-1}))/\Delta\big) + \sum_{1}^{N-1} F\big(x_k, y(x_k), (y(x_{k+1}) - y(x_{k-1}))/2\Delta\big)\Delta \tag{16.33}$$

When you keep the endpoints fixed, this is a function of $N-1$ variables, $\{y_k = y(x_k)\}$ for $1 \le k \le N-1$, and to find the minimum or maximum you simply take the partial derivative with respect to each of them. It is *not* a function of any of the $\{x_k\}$ because those are defined and fixed by the partition $x_k = a + k\Delta$. The clumsy part is keeping track of the notation. When you differentiate with respect to a particular $y_\ell$, most of the terms in the sum (16.33) don't contain it. There are only three terms in the sum that contribute: $\ell$ and $\ell \pm 1$. In the figure $N = 5$, and the $\ell = 2$ coordinate ($y_2$) is being changed. For all the indices $\ell$ except the two next to the endpoints ($1$ and $N-1$), this is

$$\frac{\partial}{\partial y_\ell} I_{\text{discrete}} = \frac{\partial}{\partial y_\ell}\Big[F\big(x_{\ell-1}, y_{\ell-1}, (y_\ell - y_{\ell-2})/2\Delta\big) + F\big(x_\ell, y_\ell, (y_{\ell+1} - y_{\ell-1})/2\Delta\big) + F\big(x_{\ell+1}, y_{\ell+1}, (y_{\ell+2} - y_\ell)/2\Delta\big)\Big]\Delta$$




An alternate standard notation for partial derivatives will help to keep track of the manipulations:

$D_1F$ is the derivative with respect to the first *argument*

The above derivative is then

$$\Big[D_2F\big(x_\ell, y_\ell, (y_{\ell+1} - y_{\ell-1})/2\Delta\big) + \frac{1}{2\Delta}\big[D_3F\big(x_{\ell-1}, y_{\ell-1}, (y_\ell - y_{\ell-2})/2\Delta\big) - D_3F\big(x_{\ell+1}, y_{\ell+1}, (y_{\ell+2} - y_\ell)/2\Delta\big)\big]\Big]\Delta \tag{16.34}$$

There is no $D_1F$ because the $x_\ell$ is essentially an index.

If you now take the limit $\Delta \to 0$, the third argument in each function returns to the derivative $y'$ evaluated at various $x_k$s:

$$\Big[D_2F\big(x_\ell, y_\ell, y'_\ell\big) + \frac{1}{2\Delta}\big[D_3F\big(x_{\ell-1}, y_{\ell-1}, y'_{\ell-1}\big) - D_3F\big(x_{\ell+1}, y_{\ell+1}, y'_{\ell+1}\big)\big]\Big]\Delta$$

$$= \Big[D_2F\big(x_\ell, y(x_\ell), y'(x_\ell)\big) + \frac{1}{2\Delta}\big[D_3F\big(x_{\ell-1}, y(x_{\ell-1}), y'(x_{\ell-1})\big) - D_3F\big(x_{\ell+1}, y(x_{\ell+1}), y'(x_{\ell+1})\big)\big]\Big]\Delta \tag{16.35}$$

Now take the limit that $\Delta \to 0$, and the last part is precisely the definition of (minus) the derivative with respect to $x$. This then becomes

$$\frac{1}{\Delta}\frac{\partial}{\partial y_\ell} I_{\text{disc}} \to D_2F(x_\ell, y_\ell, y'_\ell) - \frac{d}{dx}D_3F(x_\ell, y_\ell, y'_\ell) \tag{16.36}$$

Translate this into the notation that I've been using and you have Eq. (16.10). Why did I divide by $\Delta$ in the final step? That's the equivalent of looking for the coefficient of both the $dx$ and the $\delta y$ in Eq. (16.10). It can be useful to retain the discrete approximation of Eq. (16.34) or (16.35) to the end of the calculation. This allows you to do numerical calculations in cases where the analytic equations are too hard to manipulate.

Again, not quite yet. The two cases $\ell = 1$ and $\ell = N - 1$ have to be handled separately. You need to go back to Eq. (16.33) to see how they work out. The factors show up in different places, but the final answer is the same. See problem 16.15.

It is curious that when formulating the problem this way, you don't seem to need a partial integration. The result came out automatically. Would that be true with some integration method other other than the trapezoidal rule? See problem 16.16.

## 16.7 Classical Mechanics

The calculus of variations provides a way to reformulate Newton's equations of mechanics. The results produce efficient techniques to set up complex problems and they give insights into the symmetries of the systems. They also provide alternate directions in which to generalize the original equations.

Start with one particle subject to a force, and the force has a potential energy function $U$. Following the traditions of the subject, denote the particle's kinetic energy by $T$. Picture this first in rectangular coordinates, where $T = m\vec{v}^{\,2}/2$ and the potential energy is $U(x_1, x_2, x_3)$. The functional $S$ depends on the path $[x_1(t), x_2(t), x_3(t)]$ from the initial to the final point. The integrand is the Lagrangian, $L = T - U$.

$$S[\vec{r}\,] = \int_{t_1}^{t_2} L\big(\vec{r}, \dot{\vec{r}}\,\big)\,dt, \qquad \text{where} \qquad L = T - U = \frac{m}{2}\big(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\big) - U(x_1, x_2, x_3) \tag{16.37}$$

The statement that the functional derivative is zero is

$$\frac{\delta S}{\delta x_k} = \frac{\partial L}{\partial x_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_k}\right) = -\frac{\partial U}{\partial x_k} - \frac{d}{dt}(m\dot{x}_k)$$

Set this to zero and you have

$$m\ddot{x}_k = -\frac{\partial U}{\partial x_k}, \qquad \text{or} \qquad m\frac{d^2\vec{r}}{dt^2} = \vec{F} \tag{16.38}$$

That this integral of $L\,dt$ has a zero derivative is $\vec{F} = m\vec{a}$. Now what? This may be elegant, but does it accomplish anything? The first observation is that when you state the problem in terms of this integral it is independent of the coordinate system. If you specify a path in space, giving the velocity at each point along the path, the kinetic energy and the potential energy are well-defined at each point on the path and the integral $S$ is too. You can now pick whatever bizarre coordinate system that you want in order to do the computation of the functional derivative. Can't you do this with $\vec{F} = m\vec{a}\,$? Yes, but computing an acceleration in an odd coordinate system is a lot more work than computing a velocity. A second advantage will be that it's easier to handle constraints by this method. The technique of Lagrange multipliers from section 8.12 will apply here too.

Do the simplest example: plane polar coordinates. The kinetic energy is

$$T = \frac{m}{2}\big(\dot{r}^2 + r^2\dot{\phi}^2\big)$$

The potential energy is a function $U$ of $r$ and $\phi$. With the of the Lagrangian defined as $T-U$, the variational derivative determines the equations of motion to be

$$\begin{aligned}
S[r,\phi] &= \int_{t_1}^{t_2} L\big(r(t),\phi(t)\big)\,dt \to \\
\frac{\delta S}{\delta r} &= \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = mr\dot\phi^2 - \frac{\partial U}{\partial r} - m\ddot r = 0 \\
\frac{\delta S}{\delta \phi} &= \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} = -\frac{\partial U}{\partial \phi} - m\frac{d}{dt}\big(r^2\dot\phi\big) = 0
\end{aligned}$$

These are the components of $\vec F = m\vec a$ in polar coordinates. If the potential energy is independent of $\phi$, the second equation says that angular momentum is conserved: $mr^2\dot\phi$.

What do the discrete approximate equations (16.34) or (16.35) look like in this context? Look at the case of one-dimensional motion to understand an example. The Lagrangian is

$$L = \frac{m}{2}\dot x^2 - U(x)$$

Take the expression in Eq. (16.34) and set it to zero.

$$-\frac{dU}{dx}(x_\ell) + \frac{1}{2\Delta}\big[m(x_\ell - x_{\ell-2})/2\Delta - m(x_{\ell+2}-x_\ell)/2\Delta\big] = 0$$

or

$$m\frac{x_{\ell+2} - 2x_\ell + x_{\ell-2}}{(2\Delta)^2} = -\frac{dU}{dx}(x_\ell) \tag{16.39}$$

This is the discrete approximation to the second derivative, as in Eq. (11.12).

### 16.8 Endpoint Variation

Following Eq. (16.6) I restricted the variation of the path so that the endpoints are kept fixed. What if you don't? As before, keep terms to the first order, so that for example $\Delta t_b\,\delta y$ is out. Because the most common application of this method involves integrals with respect to time, I'll use that integration variable

$$\begin{aligned}
\Delta S &= \int_{t_a+\Delta t_a}^{t_b+\Delta t_b} dt\,L\big(t, y(t)+\delta y(t), \dot y(t) + \delta\dot y(t)\big) - \int_{t_a}^{t_b} dt\,L\big(t,y(t),\dot y(t)\big) \\
&= \int_{t_a+\Delta t_a}^{t_b+\Delta t_b} dt\,\left[L(t,y,\dot y) + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot y}\delta \dot y\right] - \int_{t_a}^{t_b} dt\,L\big(t,y(t),\dot y(x)\big) \\
&= \left[\int_{t_b}^{t_b+\Delta t_b} - \int_{t_a}^{t_a+\Delta t_a}\right] dt\,L(t,y,\dot y) + \int_{t_a}^{t_b} dt\,\left[\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot y}\delta\dot y\right]
\end{aligned}$$

Drop quadratic terms in the second line: anything involving $(\delta y)^2$ or $\delta y\delta\dot y$ or $(\delta\dot y)^2$. Similarly, drop terms such as $\Delta t_a\,\delta y$ in going to the third line. Do a partial integration on the last term

$$\int_{t_a}^{t_b} dt\,\frac{\partial L}{\partial \dot y}\frac{d\,\delta y}{dt} = \frac{\partial L}{\partial \dot y}\delta y\bigg|_{t_a}^{t_b} - \int_{t_a}^{t_b} dt\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right)\delta y(t) \tag{16.40}$$

The first two terms, with the $\Delta t_a$ and $\Delta t_b$, are to first order

$$\left[\int_{t_b}^{t_b+\Delta t_b} - \int_{t_a}^{t_a+\Delta t_a}\right] dt\,L(t,y,\dot y) = L\big((t_b,y(t_b),\dot y(t_b)\big)\Delta t_b - L\big((t_a,y(t_a),\dot y(t_a)\big)\Delta t_a$$

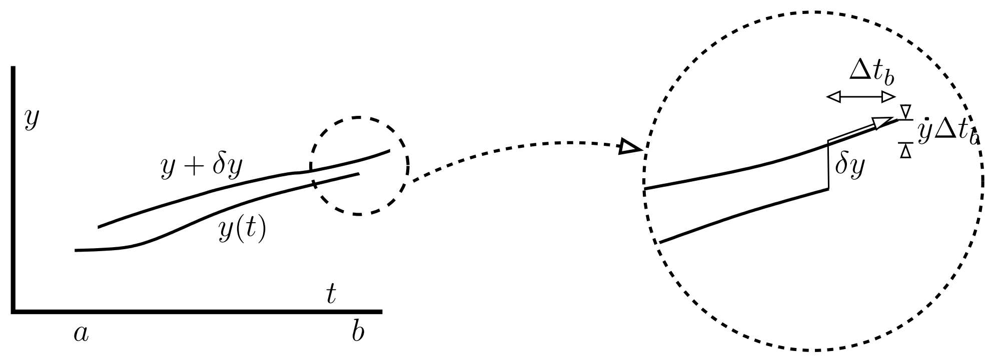
This produces an expression for $\Delta S$

$$\begin{aligned}\Delta S =& L\big((t_b, y(t_b), \dot y(t_b)\big)\Delta t_b - L\big((t_a, y(t_a), \dot y(t_a)\big)\Delta t_a \\ &+\frac{\partial L}{\partial \dot y}(t_b)\delta y(t_b) - \frac{\partial L}{\partial \dot y}(t_a)\delta y(t_a) + \int_{t_a}^{t_b} dt\,\left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right)\right]\delta y\end{aligned}\tag{16.41}$$

Up to this point the manipulation was straight-forward, though you had to keep all the algebra in good order. Now there are some rearrangements needed that are not all that easy to anticipate, adding and subtracting some terms.

Start by looking at the two terms involving $\Delta t_b$ and $\delta y(t_b)$ — the first and third terms. The change in the position of this endpoint is not simply $\delta y(t_b)$. Even if $\delta y$ is identically zero the endpoint will change in both the $t$-direction and in the $y$-direction because of the slope of the curve ($\dot y$) and the change in the value of $t$ at the endpoint ($\Delta t_b$).

The total movement of the endpoint at $t_b$ is horizontal by the amount $\Delta t_b$, and it is vertical by the amount $\big(\delta y + \dot y\Delta t_b\big)$. To incorporate this, add and subtract this second term, $\dot y\Delta t$, in order to produce this combination as a coefficient.




$$\begin{aligned}&L\big((t_b, y(t_b), \dot y(t_b)\big)\Delta t_b + \frac{\partial L}{\partial \dot y}(t_b)\delta y(t_b) \\ &\qquad= \left[L(t_b)\Delta t_b - \frac{\partial L}{\partial \dot y}(t_b)\dot y(t_b)\Delta t_b\right] + \left[\frac{\partial L}{\partial \dot y}(t_b)\dot y(t_b)\Delta t_b + \frac{\partial L}{\partial \dot y}(t_b)\delta y(t_b)\right] \\ &\qquad= \left[L - \frac{\partial L}{\partial \dot y}\dot y\right]\Delta t_b + \frac{\partial L}{\partial \dot y}\left[\dot y\Delta t_b + \delta y\right]\end{aligned}\tag{16.42}$$

Do the same thing at $t_a$, keeping the appropriate signs. Then denote

$$p = \frac{\partial L}{\partial \dot y}, \qquad H = p\dot y - L, \qquad \Delta y = \delta y + \dot y\Delta t$$

$H$ is the Hamiltonian and Eq. (16.41) becomes Noether's theorem.

$$\Delta S = \Big[p\Delta y - H\Delta t\Big]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt\,\left[\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right)\right]\delta y \tag{16.43}$$

If the equations of motion are satisfied, the argument of the last integral is zero. The change in $S$ then comes only from the translation of the endpoint in either the time or space direction. If $\Delta t$ is zero, and $\Delta y$ is the same at the two ends, you are translating the curve in space — vertically in the graph. Then

$$\Delta S = p\Delta y\Big|_{t_a}^{t_b} = \big[p(t_b) - p(t_a)\big]\Delta y = 0$$

If the physical phenomenon described by this equation is invariant under spacial translation, then momentum is conserved.

If you do a translation in time instead of space and $S$ is invariant, then $\Delta t$ is the same at the start and finish, and

$$\Delta S = \big[-H(t_b) + H(t_a)\big]\Delta t = 0$$

This is conservation of energy. Write out what $H$ is for the case of Eq. (16.37).

If you write this theorem in three dimensions and require that the system is invariant under rotations, you get conservation of angular momentum. In more complicated system, especially in field theory, there are other symmetries, and they in turn lead to conservation laws. For example conservation of charge is associated with a symmetry called "gauge symmetry" in quantum mechanical systems.

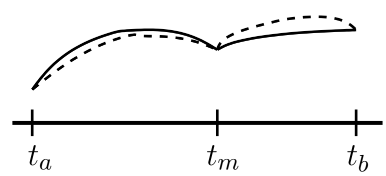
The equation (16.10), in which the variation $\delta y$ had the endpoints fixed, is much like a directional derivative in multivariable calculus. For a directional derivative you find how a function changes as the independent variable moves along some specified direction, and in the variational case the direction was specified to be with functions that were tied down at the endpoints. The development of the present section is in the spirit of finding the derivative in all possible directions, not just a special set.

### 16.9 Kinks

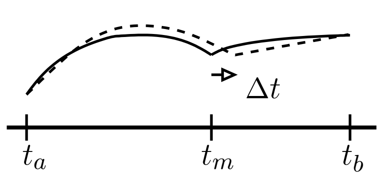
In all the preceding analysis of minimizing solutions to variational problems, I assumed that everything is differentiable and that all the derivatives are continuous. That's not always so, and it is quite possible for a solution to one of these problems to have a discontinuous derivative somewhere in the middle. These are more complicated to handle, but just because of some extra algebra. An integral such as Eq. (16.5) is perfectly well defined if the integrand has a few discontinuities, but the partial integrations leading to the Euler-Lagrange equations are *not.* You can apply the Euler-Lagrange equations only in the intervals between any kinks.

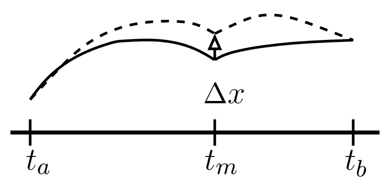
If you're dealing with a problem of the standard form $I[x] = \int_a^b dt\, L(t, x, \dot{x})$ and you want to determine whether there is a kink along the path, there are some internal boundary conditions that have to hold. Roughly speaking they are conservation of momentum and conservation of energy, Eq. (16.44), and you can show this using the results of the preceding section on endpoint variations.

$$S = \int_{t_a}^{t_b} dt\, L(t, x, \dot{x}) = \int_{t_a}^{t_m} dt\, L + \int_{t_m}^{t_b} dt\, L$$




Assume there is a discontinuity in the derivative $\dot{x}$ at a point in the middle, $t_m$. The equation to solve is still $\delta S/\delta x = 0$, and for variations of the path that leave the endpoints and the middle point alone you have to satisfy the standard Euler-Lagrange differential equations on the two segments. Now however you also have to set the variation to zero for paths that leave the endpoints alone but move the middle point.



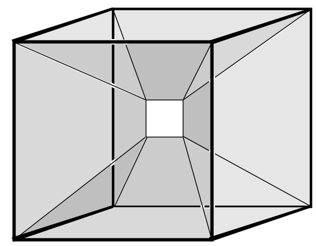

Apply Eq. (16.43) to each of the two segments, and assume that the differential equations are already satisfied in the two halves. For the sort of variation described in the last two figures, look at the endpoint variations of the two segments. They produce

$$\delta S = \Big[p\Delta x - H\Delta t\Big]_{t_a}^{t_m} + \Big[p\Delta x - H\Delta t\Big]_{t_m}^{t_b} = \big[p\Delta x - H\Delta t\big](t_m^-) - \big[p\Delta x - H\Delta t\big](t_m^+) = 0$$

These represent the contributions to the variation just above $t_m$ and just below it. This has to vanish for arbitrary $\Delta t$ and $\Delta x$, so it says

$$p(t_m^-) = p(t_m^+) \qquad \text{and} \qquad H(t_m^-) = H(t_m^+) \tag{16.44}$$

These equations, called the Weierstrass-Erdmann conditions, are two equations for the values of the derivative, $\dot{x}$, on the two side of $t_m$. The two equations for the two unknowns may tell you that there is no discontinuity in the derivative, or if there is then it will dictate the algebraic equations that the two values of $\dot{x}$ must satisfy. More dimensions means more equations of course.



There is a class of problems in geometry coming under the general heading of Plateau's Problem. What is the minimal surface that spans a given curve? Here the functional is $\int dA$, giving the area as a function of the function describing the surface. If the curve is a circle in the plane, then the minimum surface is the spanning disk. What if you twist the circle so that it does not quite lie in a plane? Now it's a tough problem. What if you have two parallel circles? Is the appropriate surface a cylinder? (No.) This subject is nothing more than the mathematical question of trying to describe soap bubbles. They're not all spheres.

Do kinks happen often? They are rare in problems that usually come up in physics, and it seems to be of more concern to those who apply these techniques in engineering. For an example that you can verify for yourself however, construct a wire frame in the shape of a cube. You can bend wire or you can make it out of solder, which is much easier to manipulate. Attach a handle to one corner so that you can hold it. Now make a soap solution that you can use to blow bubbles. (A trick to make it work better is to add some glycerine.) Now dip the wire cube into the soap and see what sort of soap film will result, filling in the surfaces among the wires. It is not what you expect, and has several faces that meet at surprising angles. There is a square in the middle. This surface has minimum area among surfaces that span the cube.

**Example**

In Eq. (16.12), looking at the shortest distance between two points in a plane, I jumped to a conclusion. To minimize the integral $\int_a^b f(y')dx$, use the Euler-Lagrange differential equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = f'(y')y'' = 0$$

This seems to say that $f(y')$ is a constant or that $y'' = 0$, implying either way that $y = Ax + B$, a straight line. Now that you know that solutions can have kinks, you have to look more closely. Take the particular example

$$f(y') = \alpha y'^4 - \beta y'^2, \qquad \text{with} \qquad y(0) = 0, \qquad \text{and} \qquad y(a) = b \tag{16.45}$$

One solution corresponds to $y'' = 0$ and $y(x) = bx/a$. Can there be others?

Apply the conditions of Eq. (16.44) at some point between $0$ and $a$. Call it $x_m$, and assume that the derivative is not continuous. Call the derivatives on the left and right $(y'^-)$ and $(y'^+)$. The first equation is

$$p = \frac{\partial L}{\partial y'} = 4\alpha y'^3 - 2\beta y', \qquad \text{and} \qquad p(x_m^-) = p(x_m^+)$$
$$4\alpha(y'^-)^3 - 2\beta(y'^-) = 4\alpha(y'^+)^3 - 2\beta(y'^+)$$
$$\big[(y'^-) - (y'^+)\big]\big[(y'^+)^2 + (y'^+)(y'^-) + (y'^-)^2 - \beta/2\alpha\big] = 0$$

If the slope is not continuous, the second factor must vanish.

$$(y'^+)^2 + (y'^+)(y'^-) + (y'^-)^2 - \beta/2\alpha = 0$$

This is one equation for the two unknown slopes. For the second equation, use the second condition, the one on $H$.

$$H = y'\frac{\partial f}{\partial y'} - f, \qquad \text{and} \qquad H(x_m^-) = H(x_m^+)$$
$$H = y'\left[4\alpha y'^3 - 2\beta y'\right] - \left[\alpha y'^4 - \beta y'^2\right] = 3\alpha y'^4 - \beta y'^2$$
$$\left[(y'^-) - (y'^+)\right]\left[(y'^+)^3 + (y'^+)^2(y'^-) + (y'^+)(y'^-)^2 + (y'^-)^3 - \beta\big((y'^+) + (y'^-)\big)/3\alpha\right] = 0$$

Again, if the slope is not continuous this is

$$(y'^+)^3 + (y'^+)^2(y'^-) + (y'^+)(y'^-)^2 + (y'^-)^3 - \beta\big((y'^+) + (y'^-)\big)/3\alpha = 0$$

These are two equations in the two unknown slopes. It looks messy, but look back at $H$ itself first. It's even. That means that its continuity will always hold if the slope simply changes sign.

$$(y'^+) = -(y'^-)$$

Can this work in the other (momentum) equation?

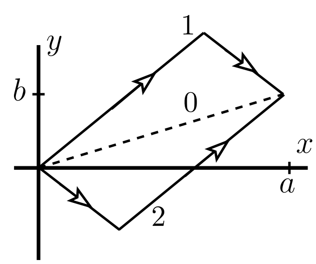
$$(y'^+)^2 + (y'^+)(y'^-) + (y'^-)^2 - \beta/2\alpha = 0 \qquad \text{is now} \qquad (y'^+)^2 = \beta/2\alpha$$

As long as $\alpha$ and $\beta$ have the same sign, this has the solution

$$(y'^+) = \pm\sqrt{\beta/2\alpha}, \qquad\qquad (y'^-) = \mp\sqrt{\beta/2\alpha} \tag{16.46}$$

The boundary conditions on the original problem were $y(0) = 0$ and $y(a) = b$. Denote $\gamma = \pm\sqrt{\beta/2\alpha}$, and $x_1 = a/2 + b/2\gamma$, then

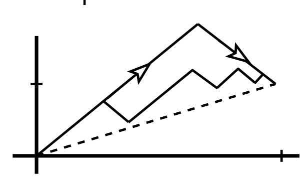
$$y = \begin{cases} \gamma x & (0 < x < x_1) \\ b - \gamma(x - a) & (x_1 < x < b) \end{cases}$$




(16.47)

The paths labeled 0, 1, and 2 are three solutions that make the variational functional derivative vanish. Which is smallest? Does that answer depend on the choice of the parameters? See problem 16.19.

Are there any other solutions? After all, once you've found three, you should wonder if it stops there. Yes, there are many — infinitely many in this example. They are characterized by the same slopes, $\pm\gamma$, but they switch back and forth several times before coming to the endpoint. The same internal boundary conditions ($p$ and $H$) apply at each corner, and there's nothing in their solutions saying that there is only one such kink.



Do you encounter such weird behavior often, with an infinite number of solutions? No, but you see from this example that it doesn't take a very complicated integrand to produce such a pathology.

### 16.10 Second Order

Except for a couple of problems in optics in chapter two, 2.35 and 2.39, I've mostly ignored the question about minimum versus maximum.

• Does it matter in classical mechanics whether the integral, $\int L\,dt$ is minimum or not in determining the equations of motion? No.

• In geometric optics, does it matter whether Fermat's principle of least time for the path of the light ray is *really* minimum? Yes, in this case it does, because it provides information about the focus.

• In the calculation of minimum energy electric potentials in a capacitor does it matter? No, but only because it's *always* a minimum.

• In problems in quantum mechanics similar to the electric potential problem, the fact that you're dealing sometimes with a minimum and sometimes not leads to some serious technical difficulties.

How do you address this question? The same way you do in ordinary calculus: See what happens to the second order terms in your expansions. Take the same general form as before and keep terms through second order. Assume that the endpoints are fixed.

$$\begin{aligned}
I[y] &= \int_a^b dx\, F\big(x, y(x), y'(x)\big) \\
\Delta I &= I[y+\delta y] - I[y] \\
&= \int_a^b dx\, F\big(x, y(x)+\delta y(x), y'(x)+\delta y'(x)\big) - \int_a^b dx\, F\big(x, y(x), y'(x)\big) \\
&= \int_a^b dx\, \left[\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial^2 F}{\partial y^2}\big(\delta y\big)^2 + 2\frac{\partial^2 F}{\partial y\partial y'}\delta y\delta y' + \frac{\partial^2 F}{\partial y'^2}\big(\delta y'\big)^2\right] \qquad (16.48)
\end{aligned}$$

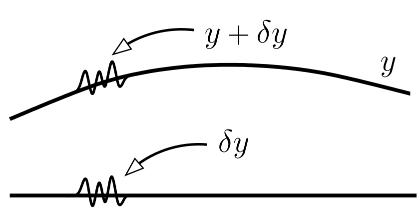
If the first two terms combine to zero, this says the first derivative is zero. Now for the next terms.

Recall the similar question that arose in section 8.11. How can you tell if a function of two variables has a minimum, a maximum, or neither? The answer required looking at the matrix of all the second derivatives of the function — the Hessian. Now, instead of a $2\times 2$ matrix as in Eq. (8.31) you have an integral.

$$\begin{aligned}
\big\langle d\vec r, H\,d\vec r\big\rangle &= (\,dx \quad dy\,)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix} dx \\ dy\end{pmatrix} \\
&\longrightarrow \int_a^b dx\,(\,\delta y \quad \delta y'\,)\begin{pmatrix} F_{yy} & F_{yy'} \\ F_{y'y} & F_{y'y'}\end{pmatrix}\begin{pmatrix} \delta y \\ \delta y'\end{pmatrix}
\end{aligned}$$

In the two dimensional case $\nabla f = 0$ defines a minimum if the product $\big\langle d\vec r, H\,d\vec r\big\rangle$ is positive for all possible directions $d\vec r$. For this new case the "directions" are the possible functions $\delta y$ and its derivative $\delta y'$.

The direction to look first is where $\delta y'$ is big. The reason is that I can have a very small $\delta y$ that has a very big $\delta y'$: $10^{-3}\sin\big(10^6 x\big)$. If $\Delta I$ is to be positive in every direction, it has to be positive in this one. That requires $F_{y'y'} > 0$.



Is it really that simple? No. First the $\delta y$ terms can be important too, and second $y$ can itself have several components. Look at the latter first. The final term in Eq. (16.48) should be

$$\int_a^b dx\, \frac{\partial^2 F}{\partial y'_m \partial y'_n} \delta y'_m \delta y'_n$$

This set of partial derivatives of $F$ is *at each point along the curve* a Hessian. At each point it has a set of eigenvalues and eigenvectors, and if all along the path all the eigenvalues are always positive, it meets the first, necessary conditions for the original functional to be a minimum. If you look at an example from mechanics, for which the independent variable is time, these $y'_n$ terms are then $\dot{x}_n$ instead. Terms such as these typically represent kinetic energy and you expect that to be positive.

An example:

$$S[\vec{r}\,] = \int_0^T dt\, L(x, y, \dot{x}, \dot{y}, t) = \int_0^T dt\, \frac{1}{2}\big[\dot{x}^2 + \dot{y}^2 + 2\gamma t \dot{x}\dot{y} - x^2 - y^2\big]$$

This is the action for a particle moving in two dimensions $(x, y)$ with the specified Lagrangian. The equation of motion are

$$\frac{\delta S}{\delta x} = -x - \ddot{x} - \gamma(t\ddot{y} + \dot{y}) = 0$$
$$\frac{\delta S}{\delta y} = -y - \ddot{y} - \gamma(t\ddot{x} + \dot{x}) = 0$$

If $\gamma = 0$ you have two independent harmonic oscillators.

The matrix of derivatives of $L$ with respect to $\dot{x} = \dot{y}_1$ and $\dot{y} = \dot{y}_2$ is

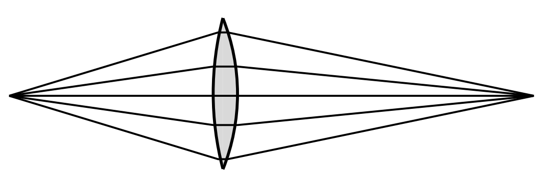
$$\frac{\partial^2 L}{\partial \dot{y}_m \partial \dot{y}_n} = \begin{pmatrix} 1 & \gamma t \\ \gamma t & 1 \end{pmatrix}$$

The eigenvalues of this matrix are $1 \pm \gamma t$, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This Hessian then says that $S$ should be a minimum up to the time $t = 1/\gamma$, but not after that. This is also a singular point of the differential equations for $x$ and $y$.

**Focus**

When the Hessian made from the $\delta y'^2$ terms has only positive eigenvalues everywhere, the preceding analysis might lead you to believe that the functional is always a minimum. Not so. That condition is necessary; it is not sufficient. It says that the functional is a minimum with respect to rapidly oscillating $\delta y$. It does not say what happens if $\delta y$ changes gradually over the course of the integral. If this happens, and if the length of the interval of integration is long enough, the $\delta y'$ terms may be the small ones and the $(\delta y)^2$ may then dominate over the whole length of the integral. This is exactly what happens in the problems 2.35, 2.39, and 16.17.

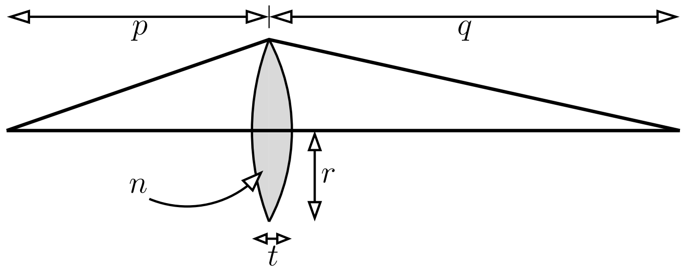
When this happens in an optical system, where the functional $T = \int d\ell/v$ is the travel time along the path, it signals something important. You have a focus. An ordinary light ray obeys Fermat's principle that $T$ is stationary with respect to small changes in the path. It is a minimum if the path is short enough. A focus occurs when light can go from one point to another by many different paths, and for still longer paths the path is neither minimum nor maximum.



In the integral for $T$, where the starting point and the ending point are the source and image points, the second order variation will be zero for these long, gradual changes in the path. The straight-line path through the center of the lens takes *least* time if its starting point and ending point are closer than this source and image. The same path will be a saddle (neither maximum nor minimum) if the points are farther apart than this. This sort of observation led to the development of the mathematical subject called "Morse Theory," a topic that has had applications in studying such diverse subjects as nuclear fission and the gravitational lensing of light from quasars.

**Thin Lens**

This provides a simple way to understand the basic equation for a thin lens. Let its thickness be $t$ and its radius $r$.




Light that passes through this lens along the straight line through the center moves more slowly as it passes through the thickness of the lens, and takes a time

$$T_1 = \frac{1}{c}(p + q - t) + \frac{n}{c}t$$

Light that take a longer path through the edge of the lens encounters no glass along the way, and it takes a time

$$T_2 = \frac{1}{c}\Big[\sqrt{p^2 + r^2} + \sqrt{q^2 + r^2}\Big]$$

If $p$ and $q$ represent the positions of a source and the position of its image at a focus, these two times should be equal. At least they should be equal in the approximation that the lens is thin and when you keep terms only to the second order in the variation of the path.

$$T_2 = \frac{1}{c}\Big[p\sqrt{1 + r^2/p^2} + q\sqrt{1 + r^2/q^2}\Big] = \frac{1}{c}\Big[p\big(1 + r^2/2p^2\big) + q\big(1 + r^2/2q^2\big)\Big]$$

Equate $T_1$ and $T_2$.

$$\begin{aligned} (p + q - t) + nt &= \Big[p\big(1 + r^2/2p^2\big) + q\big(1 + r^2/2q^2\big)\Big] \\ (n-1)t &= \frac{r^2}{2p} + \frac{r^2}{2q} \\ \frac{1}{p} + \frac{1}{q} &= \frac{2(n-1)t}{r^2} = \frac{1}{f} \end{aligned} \tag{16.49}$$

This is the standard equation describing the focusing properties of thin lenses as described in every elementary physics text that even mentions lenses. The focal length of the lens is then $f = r^2/2(n-1)t$. That is *not* the expression you usually see, but it is the same. See problem 16.21. Notice that this equation for the focus applies whether the lens is double convex or plano-convex or meniscus: . If you allow the thickness $t$ to be negative (equivalent to saying that there's an extra time delay at the edges instead of in the center), then this result still works for a diverging lens, though the analysis leading up to it requires more thought.

## Exercises

**1** For the functional $F[x] = x(0) + \int_0^\pi dt\,\big(x(t)^2 + \dot{x}(t)^2\big)$ and the function $x(t) = 1 + t^2$, evaluate $F[x]$.

**2** For the functional $F[x] = \int_0^1 dt\,x(t)^2$ with the boundary conditions $x(0) = 0$ and $x(1) = 1$, what is the minimum value of $F$ and what function $x$ gives it? Start by drawing graphs of various $x$ that satisfy these boundary conditions. Is there any reason to require that $x$ be a continuous function of $t$?

**3** With the functional $F$ of the preceding exercise, what is the functional derivative $\delta F/\delta x$?

## Problems

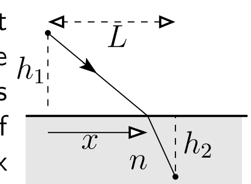
**16.1** You are near the edge of a lake and see someone in the water needing help. What path do you take to get there in the shortest time? You can run at a speed $v_1$ on the shore and swim at a probably slower speed $v_2$ in the water. Assume that the edge of the water forms a straight line, and express your result in a way that's easy to interpret, not as the solution to some quartic equation. Ans: Snell's Law.

**16.2** The cycloid is the locus of a point that is on the edge of a circle that is itself rolling along a straight line — a pebble caught in the tread of a tire. Use the angle of rotation as a parameter and find the parametric equations for $x(\theta)$ and $y(\theta)$ describing this curve. Show that it is Eq. (16.17).

**16.3** In Eq. (16.17), describing the shortest-time slide of a particle, what is the behavior of the function for $y \ll a$? In figuring out the series expansion of $w = \cos^{-1}(1-t)$, you may find it useful to take the cosine of both sides. Then you should be able to find that the two lowest order terms in this expansion are $w = \sqrt{2t} - t^{3/2}/12\sqrt{2}$. You will need both terms. Ans: $x = \sqrt{2y^3/a}\,/3$

**16.4** The dimensions of an ordinary derivative such as $dx/dt$ is the quotient of the dimensions of the numerator and the denominator (here L/T). Does the same statement apply to the functional derivative?

**16.5** Use Fermat's principle to derive both Snell's law and the law of reflection at a plane surface. Assume two straight line segments from the starting point to the ending point and minimize the total travel time of the light. The drawing applies to Snell's law, and you can compute the travel time of the light as a function of the coordinate $x$ at which the light hits the surface and enters the higher index medium.




**16.6** Analyze the path of light over a roadway starting from Eq. (16.23) but using $x$ as the independent variable instead of $y$.

**16.7** (**a**) Fill in the steps leading to Eq. (16.31). And do you understand the point of the rearrangements that I did just preceding it? Also, can you explain why the form of the function Eq. (16.30) should have been obvious without solving any extra boundary conditions? (**b**) When you can explain that in a few words, then what general cubic polynomial can you use to get a still better result?

**16.8** For the function $F(x,y,y') = x^2 + y^2 + y'^2$, explicitly carry through the manipulations leading to Eq. (16.41).

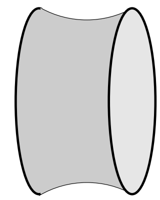
**16.9** Use the explicit variation in Eq. (16.8) and find the minimum of that function of $\epsilon$. Compare that minimum to the value found in Eq. (16.11). Ans: 1.64773

**16.10** Do either of the functions, Eqs. (16.29) or (16.30), satisfy Laplace's equation?

**16.11** For the function $F(x,y,y') = x^2 + y^2 + y'^2$, repeat the calculation of $\delta I$ only now keep all the higher order terms in $\delta y$ and $\delta y'$. Show that the solution Eq. (16.11) is a minimum.

**16.12** Use the techniques leading to Eq. (16.21) in order to solve the brachistochrone problem Eq. (16.13) again. This time use $x$ as the independent variable instead of $y$.

**16.13** On a right circular cylinder, find the path that represents the shortest distance between two points. $d\ell^2 = dz^2 + R^2 d\phi^2$.



**16.14** Put two circular loops of wire in a soap solution and draw them out, keeping their planes parallel. If they are fairly close you will have a soap film that goes from one ring to the other, and the minimum energy solution is the one with the smallest area. What is the shape of this surface? Use cylindrical coordinates to describe the surface. It is called a catenoid, and its equation involves a hyperbolic cosine.

**16.15** There is one part of the derivation going from Eq. (16.33) to (16.36) that I omitted: the special cases of $\ell = 1$ and $\ell = N - 1$. Go back and finish that detail, showing that you get the same result even in this case.

**16.16** Section 16.6 used the trapezoidal rule for numerical integration and the two-point centered difference for differentiation. What happens to the derivation if (**a**) you use Simpson's rule for integration or if (**b**) you use an asymmetric differentiation formula such as $y'(0) \approx [y(h) - y(0)]/h$?

**16.17** For the simple harmonic oscillator, $L = m\dot{x}^2/2 - m\omega^2 x^2/2$. Use the time interval $0 < t < T$ so that $S = \int_0^T L\,dt$, and find the equation of motion from $\delta S/\delta x = 0$. When the independent variable $x$ is changed to $x + \delta x$, keep the second order terms in computing $\delta S$ this time and also make an explicit choice of

$$\delta x(t) = \epsilon \sin(n\pi t/T)$$

For integer $n = 1, 2 \ldots$ this satisfies the boundary conditions that $\delta x(0) = \delta x(T) = 0$. Evaluate the change is $S$ through second order in $\epsilon$ (that is, do it exactly). Find the conditions on the interval $T$ so that the solution to $\delta S/\delta x = 0$ is in fact a minimum. Then find the conditions when it isn't, and what is special about the $T$ for which $S[x]$ changes its structure? Note: This $T$ is defined independently from $\omega$. It's specifies an arbitrary time interval for the integral.

**16.18** Eq. (16.37) describes a particle with a specified potential energy. For a charge in an electromagnetic field let $U = qV(x_1, x_2, x_3, t)$ where $V$ is the electric potential. Now how do you include magnetic effects? Add another term to $L$ of the form $C\dot{\vec{r}} \cdot \vec{A}(x_1, x_2, x_3, t)$. Figure out what the Lagrange equations are, making $\delta S/\delta x_k = 0$. What value must $C$ have in order that this matches $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}\,) = m\vec{a}$ with $\vec{B} = \nabla \times \vec{A}$? What is $\vec{E}$ in terms of $V$ and $\vec{A}$? Don't forget the chain rule. Ans: $C = q$ and then $\vec{E} = -\nabla V - \partial\vec{A}/\partial t$

**16.19** (**a**) For the solutions that spring from Eq. (16.46), which of the three results shown have the largest and smallest values of $\int f\,dx$? Draw a graph of $f(y')$ and see where the characteristic slope of Eq. (16.46) is with respect to the graph.
(**b**) There are circumstances for which these kinked solutions, Eq. (16.47) do and do not occur; find them and explain them.

**16.20** What are the Euler-Lagrange equations for $I[y] = \int_a^b dx\, F(x, y, y', y'')$?

**16.21** The equation for the focal length of a thin lens, Eq. (16.49), is not the traditional one found in most texts. That is usually expressed in terms of the radii of curvature of the lens surfaces. Show that this is the same. Also note that Eq. (16.49) is independent of the optics sign conventions for curvature.

**16.22** The equation (16.25) is an approximate solution to the path for light above a hot road. Is there a function $n = f(y)$ representing the index of refraction above the road surface such that this equation would be its exact solution?

**16.23** On the first page of this chapter, you see the temperature dependence of length measurements.
(**a**) Take a metal disk of radius $a$ and place it centered on a block of ice. Assume that the metal reaches an equilibrium temperature distribution $T(r) = T_0(r^2/a^2 - 1)$. The temperature at the edge is $T = 0$, and the ruler is calibrated there. The disk itself remains flat. Measure the distance from the origin straight out to the radial coordinate $r$. Call this measured radius $s$. Measure the circumference of the circle at this radius and then express this circumference in terms of the measured radius $s$.
(**b**) On a sphere of radius $R$ (constant temperature) start at the pole ($\theta = 0$) and write the distance along the arc at constant $\phi$ down to the angle $\theta$. Now go around the circle at this constant $\theta$ and write its circumference. Express this circumference in terms of the distance you just wrote for the "radius" of this circle.
(**c**) Show that the geometry you found in (**a**) is the same as that in (**b**) and find the radius of the sphere that this "heat metric" expresses. Ans: $R = a/2\sqrt{\alpha T_0(1 - \alpha T_0)} \approx a/2\sqrt{\alpha T_0}$

**16.24** Using the same techniques as in section 16.5, apply these methods to two concentric spheres. Again, use a linear and then a quadratic approximation. Before you do this, go back to Eq. (16.30) and see if you can arrive at that form directly, *without* going through all the manipulations of solving for $\alpha$ and $\beta$. That is, determine how you could have gotten to (16.30) easily. Check some numbers against the exact answer.

**16.25** For the variational problem Eq. (16.45) one solution is $y = bx/a$. Assume that $\alpha,\, \beta > 0$ and determine if this is a minimum or maximum or neither. Do this also for the other solution, Eq. (16.47). Ans: The first is min if $b/a > \sqrt{\beta/6\alpha}$. The kinked solution is always a minimum.

**16.26** If you can construct glass with a variable index of refraction, you can make a flat lens with an index that varies with distance from the axis. What function of distance must the index $n(r)$ be in order that this flat cylinder of glass of thickness $t$ has a focal length $f$? All small angles and thin lenses of course. Ans: $n(r) = n(0) - r^2/2ft$.